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Theory of the Measurement of Wind by Shooting Spheres Upward

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IX. *Theory of the Measurement of Wind by Shooting Spheres Upward.*

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Communicated by Dr. G. C. Simpson, F.R.S.

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Contents.

§ 1. Introduction.

For a purpose such as numerical prediction by finite differences, meteorological observations are useless if they are not very complete. The most complete sets of observations are those which have been made on "international balloon days," and if one looks at the charts of them issued by V. Bjerknes and R. Wenger from Leipzig* one notices the conspicuous gaps where clouds have prevented the measurement of the upper wind

* 'Leipzig Geophys. Instit.'—"Synoptische Darstellungen."

 [Published, July 16, 1923.

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by means of pilot balloons. More wind observations above clouds would, no doubt, be useful in ways other than that mentioned above. Some have been obtained recently by sound-ranging or by sending instruments up on kite-balloons.

The chief advantages of this new method of observing are: (i) that it can be employed when there is fog or low cloud which would hide a pilot balloon; (ii) that it is much cheaper than sound-ranging or than captive ballooning at least for the height of 650 metres to which experiments have so far extended. Minor advantages are (iii) that the steel sphere is much less affected than a pilot balloon by vertical air currents; (iv) that these projectiles give the wind almost directly overhead, while pilot balloons drift far horizontally from the nominal site of the observation; (v) that the operations are easily performed at night; (vi) that it is well suited to making repeated observations of wind at short intervals of time, either in order to study the variations, or in order to remove them by averaging.*

The maximum height attained with the present apparatus has only been about 650 metres, and in that respect pilot balloons are far superior in clear weather. But an increase of height appears to be quite possible.

A single observer can make the observation in about half an hour. The material consumed costs, very roughly, 1*s*. for determinations up to 300 metres, or 4*s*. up to 650 metres.

A full description of the practical aspect of the measurement is being published by the Meteorological Office in their Professional Note No. 34, so that it should suffice here to describe the practice very briefly.

The spheres are the admirably accurate steel balls used in engineering. The gun is not rifled. The spheres are shot upward in a direction which is slightly inclined from the vertical, and which is adjusted by trial until the returning ball falls quite close to the shelter which protects both gun and observer. The tilt of the gun-barrel is recorded by writing down the cosines of the angles which the axis of the barrel makes with two horizontal lines drawn east-west and north-south. These two components of tilt are read directly on a special scale. In the following theory these components of tilt are denoted by β or by β_X, β_Y where distinction between the x and y components is necessary. The time of absence of each shot is also observed by a stop-watch. In the theory this time is denoted by t'. The height to which the shot has ascended is a function of this time of absence, as will be shown.

The distance of the point of fall of the shot from the gun is estimated, usually by ear, and corrections are applied to adjust the two components of tilt to what they would have been if the ball on its return had hit the muzzle. The tilt, so corrected, is called the "balancing tilt" because it balances the wind.

The same size of sphere which was first sent up for a short time of absence of, say, 9 seconds, is now shot in succession with gradually increasing times of absence, ranging for the present apparatus up to 25 seconds, and for each shot the components of tilt,

* As proposed in 'Weather Prediction by Numerical Process,' Camb. Press, Chapter X.

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after correction for the point of fall, are plotted on a diagram against the time of absence.

The gustiness of the wind produces a scatter of the observed points on the diagram. This scatter is ignored by drawing a smooth curve through or among the points.

Thus the observation yields two smooth curves giving the components of balancing tilt β_X, β_Y each as a function of the time of absence t'. We know also the diameter and mass of the sphere. The problem now before us is from these data to extract the wind components as functions of the height.

Before engaging upon this problem it may be well to mention, that we need a certain aerodynamic constant, which has been made the subject of a study, an account of which may be published elsewhere.

It should also be mentioned that a few of the meteorological results obtained by this method have already been published.*

The average tilt of the gun is about one-tenth of a radian from the vertical. So that the vertical component of the motion in a wind is nearly the same as it would be in still air. Much use is made of this approximation.

As the projectile is spherical, the force of resistance is parallel to the relative velocity of the projectile and the air.

The ascending or descending speed of the sphere is much greater than the horizontal wind speed except just near the top of the flight. Hence the total resistance of the air upon the sphere is determined almost entirely by the vertical motion, which is known. The effect of the horizontal wind is mainly to tilt the resistance-vector through a small angle, without appreciably altering its magnitude. A consequence, of great practical convenience, is that the tilt of the gun which brings the bullet back on itself is simply proportional to the wind-speed, if the latter is uniform; or if the speed varies with height, the tilt is a homogeneous linear function of the wind-speeds in the successive layers.† Another consequence is that the effects of wind-components at right angles are independent of one another. These are approximations which improve as the wind velocity diminishes, or as the size of the projectile increases. For instance, the terminal velocity of the steel ball of 0·8334 cm. diameter is about 40 m/s., so that in the latter part of its descent a wind of 10 m/s. increases the resultant speed relative to the air to $\sqrt{40^2+10^2}$ that is by 3 per cent. For the larger ball used in the fowling piece the terminal velocity is about 60 m/s. and the corresponding increase only 1·5 per cent. A linear theory is presented now. When time permits it should be improved by taking account of these quadratic adjustments.

§ 2. Notation.

Take rectangular co-ordinate axes Ox, Oy, Oh of which Oh points upwards. Let their origin be at the gun.

* 'Quart. Journ. R. Meteor. Soc.,' January, 1923, p. 34.

† This may be evident. It is assumed in § 3; but proved in detail for the square law of resistance in § 4.4.

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Let t be the time reckoned forward from the vertex which is the highest point of the trajectory.

Let s be the length of path also reckoned forward from the vertex so that s and t are negative in the ascent, positive in the descent. The origin of s thus differs from that of h.

Let r_X, r_Y, r_H be the component velocities of the sphere in the directions, x increasing, y increasing and upwards, reckoned relative to the air through which the sphere is momentarily passing. It is convenient to reckon s also relative to the air so that

$$ds = dt\sqrt{(r_X{}^2+r_Y{}^2+r_H)}.$$

Following a common practice let us denote $\sqrt{(r_X{}^2+r_Y{}^2+r_H{}^2)}$ by r, and reckon r always positive

$$\text{thus } ds = r\,dt.$$

Let nr^2 be the negative acceleration produced in a sphere of this particular size and density when moving with velocity r by the resistance of the air alone, so that n is a quantity which varies indeed with r, but varies less rapidly than nr^2.

Let g be the acceleration of gravity.

Let t' be the time of absence of the ball from leaving the muzzle to hitting the gun.

Let l be the maximum height attained by the sphere above the ground. Then, since the trajectory is nearly vertical, and since we take the origin of the path s at the vertex, it follows that the sphere leaves the gun at $s=-l$ and strikes the ground at $s = +l$ approximately.

It is convenient to use $-l$ and $+l$ as suffixes to other quantities. Thus t_{-l} is the instant at which the sphere leaves the gun. It is negative because we take the origin of time when the sphere is at the vertex. So the duration of ascent, being positive, is $-t_{-l}$. Similarly the instant at which the sphere falls on the ground is t_{+l}, which is positive and equal to the duration of descent.

Where suffixes cannot be used let us put

$$a = -t_{-l}\,; \quad b = t_{+l}.$$

The time of absence $= t' = a+b = t_l - t_{-l}$.

Let β_X, β_Y be the cosines of the angles between the barrel and the axes of x and y.

§ 3. From the Range-in-still-air, which can be Measured, to Deduce the Tilt which will Balance a Wind that is Independent of Height.

This detached problem is placed at the beginning, because it can be treated with considerable generality by the aid of elementary algebra.

We refer, as always, to nearly vertical flight, so that, as explained in § 2 above,

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the wind velocity is proportional to the tilt which it balances, or to the drift which it produces in the absence of tilt ; and further these effects are superposable.

It is supposed in the four following cases that the muzzle velocity is the same for all.

Case A.—Wind velocity v_X. A fixed gun pointed Vertically at $x = 0$.

Ball strikes ground at $x = cv_X$, where c is independent of v_X, but unknown as yet. (1)

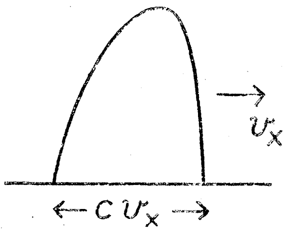

Fig. 1.—Case A.

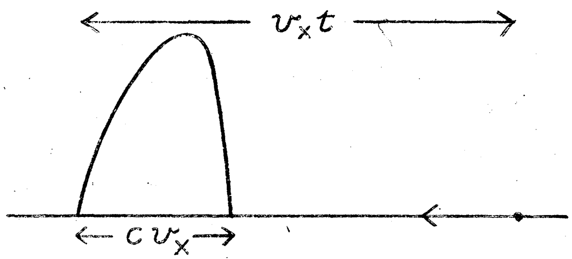

Fig. 2.—Case B.

Case B.—Air Still. Gun Carried on Motor-car with Horizontal Velocity $-v_X$ and pointed Vertically.

Then, relative to the moving gun, the path of the ball is exactly the same as in Case A. The same is true relative to the air. Therefore the bullet strikes the ground at a distance cv_X from the simultaneous position of the gun. How far is this from the position of the gun at the moment of firing ?

Let t' be time of absence of the ball. Then during t' the gun moves a distance $-v_X t'$. So the shot falls at a distance

$$-v_X t' + cv_X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

from the point where the gun was when it was fired.

Case C.—Air Still. Fixed Gun Tilted.

Now we should have given almost the same initial motion to the ball, relative to the still air, if instead of pointing the gun vertically and carrying it horizontally with a velocity $-v_X$ we had kept the gun fixed and tilted it through an angle to give the ball a horizontal component $-v_X$ relative to the air or ground. If u be the muzzle velocity of the ball, and if the cosine of the angle between the barrel and the x axis be β, then β must be such that $u\beta = -v_X$. The equivalence of tilt to translation holds provided β is small compared with unity. In the observations β averaged about $0 \cdot 1$.

Now on comparing Case B with Case C we see that in the latter the shot must fall at a horizontal distance $-(v_X t' - cv_X)$ from the gun. This distance, which is the range in still air, is seen on inserting the value of v_X to be

$$u\beta\,(t' - c). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

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As an illustration, the use of this formula, c, for the Daisy Air-gun shooting 0·437 cm. Hoffman steel balls, was determined by Mr. L. H. G. DINES and the writer on a calm day, by giving tilts of a few degrees from the vertical to the four points of the compass, and noting where the sphere fell. The mean of sixteen such experiments showed that 1 degree of tilt corresponded to a range of 5·5 metres. So as $t' = 10 \cdot 1$ seconds, and as the muzzle speed u was found by a ballistic pendulum to be 123 metres per second, it follows from (3) that

$$c = 7 \cdot 5 \text{ seconds.}$$

Case D.—Wind Independent of Height. Fixed Gun Tilted.

Now let there be superposed on the phenomena of Case C a wind velocity v_X' which just produces an equal and opposite range, so that the returning ball hits the gun.

The range produced by v_X' alone would be cv_X' as in Case A. The range produced by tilt alone would be $u\beta\,(t'-c)$. The sum of these, namely, $cv_X'+u\beta\,(t'-c)$ is by hypothesis zero. Therefore

$$v_X' = -u\beta\,\frac{t'-c}{c}. \qquad (4)$$

This formula enables us to measure the wind v_X' if we know the muzzle velocity u, the tilt which gives β, the time of absence t' and c. But c has just been determined.

Thus, taking again the Daisy Air-gun as illustration, formula (4) becomes, when $c = 7 \cdot 5$ seconds

$$v'_X = -4300\beta,$$

which in words reads: if the tilt be reckoned from the vertical in radians and if we neglect the difference between a small angle and its sine, then (the speed of the uniform wind) = (the balancing tilt)×(43 metres per second).

Actual winds are not independent of height. Eighteen comparisons with the DINES anemometer which is placed at a height of 26 metres gave the following mean result:—(the wind speed at the anemometer) = (the balancing tilt)×(21 metres per second).

As the ball went up about 120 metres, it is to be expected that the speed at the anemometer would be much less than that of the "equivalent uniform wind," and the numbers 43 and 21 show that it is so.*

The theory of this section makes no reference to any particular law of resistance and so is true, with the approximations mentioned, for all. Thus it may perhaps lead on to the treatment of winds which vary with height when the law of resistance is complicated. The rest of this paper, however, deals chiefly with resistances proportional to the square of the velocity.

* Hellmann observed the wind-speeds at heights of 120 m. and 32 m. and found a mean ratio of 1·30 between them. 'Sitz. K. P. Akad.,' Berlin, 1917, X.

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§ 4. General Theory assuming that the Force of Resistance is Proportional to the Square of the Velocity.

§ 4.1. *Introduction.*

This theory has the very great advantage that it extends immediately to any size of sphere, and that the formulæ are simple.

It is not applicable to a range of velocity which includes the speed of sound, 330 m/s. But the velocities used in the observations have mostly lain in the range 0 to 300 m/s. The parts of the theory will be taken in the following order. First, the constants will be simplified by the introduction of " natural " units. Next, the relations between muzzle-velocity, height attained, and time of absence will be computed for vertical motion in still air. We shall then be ready for the main problem which is :—Given the balancing tilt as a function of the time of absence, required to determine the wind in all the levels traversed. This problem leads to a " linear integral equation of the first kind."

§ 4.2. *Natural Units for the Square Law of Resistance.*

For the range of speeds with which the observations have been made, it is permissible to assume that the resistance of the air to the sphere is approximately equal to

$$(\text{diameter})^2 \, . \, (\text{density of air}) \, . \, (\text{velocity})^2 \, . \, \phi_1 \text{ dynes,}$$

where ϕ_1 is a numerical constant. A discussion, not yet published, shows that the value $\phi_1 = 0 \cdot 168$ is appropriate.

Knowing the diameter of the sphere and its mass we can then immediately find the terminal velocity k which it will attain when dropped from a sufficient height. This velocity k and the acceleration of gravity g suffice to define the dynamical properties of the sphere in air of the given mean temperature and pressure. By choosing units based on k and g all the particulars which we shall require concerning the motion may be very simply expressed in a form applicable to any size of sphere.

Thus let :—

$$\left.\begin{array}{l}\text{The unit of velocity be } k,\\ \text{The unit of acceleration be } g,\\ \text{The unit of time be } k/g,\\ \text{The unit of length be } k^2/g.\end{array}\right\} \quad \ldots \ldots \ldots \quad (1)$$

Quantities expressed in these natural units will be described as " naturalized quantities " and their symbols will be printed in capitals. They are " pure numbers."

Since r_H denotes the upward component of the sphere's velocity relative to the air, the corresponding " naturalized velocity " is

$$R_H = r_H/k. \quad \ldots \ldots \ldots \ldots \quad (2)$$

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Similarly, as t is the time in seconds, the "naturalized time" is

$$T = gt/k. \qquad (3)$$

The equation of vertical motion

$$\frac{dr_H}{dt} = \mp g \frac{r_H^2}{k^2} - g \qquad (4)$$

then becomes

$$dR_H/dT = \mp R_H^2 - 1, \qquad (5)$$

which is simpler than (4).

By treating k as a constant we here neglect the variation with height of the density of the air. This defect should be removed when way opens, but to attempt the removal now would over-complicate an already complicated theory.

§ 4.3. Relations between Time of Absence, Muzzle-velocity and Height Attained.

We require the following results, most of which are already known.

In the *ascent* the minus sign must be taken in (5) of the preceding section. On integrating* this equation, inserting U for the value of the naturalized muzzle-velocity, and setting $-T_{-L}$ for the whole naturalized time of ascent from a naturalized depth L before the vertex, it is found that

$$-T_{-L} = \tan^{-1} U \qquad (6)$$

or identically

$$U = \tan(-T_{-L}) \qquad (6A)$$

Again if L be the whole naturalized height attained

$$L = \tfrac{1}{2} \log_e (1 + U^2), \qquad (7)$$

$$L = \log_e \sec T_{-L}. \qquad (8)$$

In the *descent* the plus sign must be taken in (5). Then if T_L is the naturalized time of descent, and W the naturalized velocity attained, after a naturalized depth of L beyond the vertex it is found* that

$$T_L = \tfrac{1}{2} \log_e \frac{1+W}{1-W} \qquad (9)$$

or identically

$$W = \tanh T_L \qquad (9A)$$

$$L = -\tfrac{1}{2} \log_e (1 - W^2) \qquad (10)$$

And

$$L = \log \cosh T_L \qquad (11)$$

* *See* for example the text-books on dynamics by Tait and Steel, § 211, or by Lamb, Article 97.

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Eliminating L between (11) and (7) we obtain the following connection between the time of *descent* and the muzzle-velocity

$$U = \sinh T_L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Certain observations are concerned with the time of absence of a sphere projected vertically from the ground, and the relation of this time to the height attained and to the muzzle-velocity. The corresponding equations between the naturalized quantities are obtained from the foregoing, and read as follows :—

$$T' = -T_{-L} + T_L = \sec^{-1} e^L + \cosh^{-1} e^L, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$T' = -T_{-L} + T_L = \tan^{-1} U + \sinh^{-1} U. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

From these equations the following tables have been calculated.

It is remarkable how little effect the air has upon the relation of t' to l. In other words, the formula $t' = \sqrt{8l/g}$, which is valid *in vacuo*, is still approximately true in air, being only 6 per cent. in error at L = 3.

TABLE I.

U $\equiv u/k$.	T′ $\equiv gt'/k$.
0	0·000
1	1·667
2	2·551
3	3·067
4	3·421
5	3·686
6	3·898
7	4·073
8	4·223
9	4·354
10	4·469

TABLE II.

T′ $\equiv gt'/k$.	L $\equiv gl/k^2$.
0·000	0·0
1·266	0·2
1·791	0·4
2·197	0·6
2·543	0·8
2·851	1·0
3·135	1·2
3·399	1·4
3·650	1·6
3·891	1·8
4·124	2·0
4·350	2·2
4·571	2·4
4·788	2·6
5·002	2·8
5·213	3·0

§ 4.4. *Motion in a Wind which varies with Height.*

We assume that the retardation due to the air-resistance is n times the square of the velocity of the sphere relative to the air. The resultant retardation is, therefore, $n(r_X^2+r_Y^2+r_H^2)$, and, since the projectile is a sphere, the direction of the retardation is that of the vector r_X, r_Y, r_H, and its x component is

$$n(r_X^2+r_Y^2+r_H^2) \,.\, r_X/\sqrt{r_X^2+r_Y^2+r_H^2}.$$

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Thus if $\ddot{x}$, $\ddot{y}$, $\ddot{h}$ represent accelerations of the sphere relative to the earth, we have for the three dynamical equations

$$\ddot{x} = -nr_X\sqrt{r_X^2+r_Y^2+r_H^2}, \quad (1)$$

$$\ddot{y} = -nr_Y\sqrt{r_X^2+r_Y^2+r_H^2}, \quad (2)$$

$$\ddot{h} = -nr_H\sqrt{r_X^2+r_Y^2+r_H^2}-g, \quad (3)$$

in which only the positive value of the square root must be taken. Since the motion of ordinary winds is practically horizontal, the upward velocity or upward acceleration relative to the air is practically the same as that relative to the ground, so that in symbols

$$\ddot{h} = dr_H/dt; \quad \dot{h} = r_H. \quad (4)$$

Now s is the distance along the curved path, reckoned zero at the vertex, and increasing with time; and s is reckoned relative to the air, so we have

$$\dot{s} \equiv \frac{ds}{dt} = \sqrt{r_X^2+r_Y^2+r_H^2} \quad (5)$$

and the dynamical equations simplify to

$$\ddot{x} = -nr_X\dot{s}, \quad (6)$$

$$\ddot{y} = -nr_Y\dot{s}, \quad (7)$$

$$\ddot{h} = -nr_H\dot{s}-g, \quad (8)$$

in which in contrast to (1), (2) and (3) there is no need for a special statement about sign. The equations are much simplified by the fact that the upward speed is usually much greater than the horizontal relative speed. For example, in a uniform wind of 10 m/s. the steel ball, the size of a pea, is shot up with a muzzle speed of 150 m/s. at a tilt of 0·15 radian and reaches the earth again with a speed of 37 m/s.

So as the ball emerges from the muzzle

$$\sqrt{r_X^2+r_Y^2+r_H^2} = r_H(1+0{\cdot}02).$$

Again as the ball approaches the earth it is drifting somewhat with the wind and

$$\sqrt{r_X^2+r_Y^2+r_H^2} = r_H(1+\text{less than } 0{\cdot}04).$$

If we neglect the small excess of these coefficients above unity and put simply

$$\sqrt{r_X^2+r_Y^2+r_H^2} = r_H = \pm\dot{h}, \quad (9)$$

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we thereby assume that any portion of the path relative to the air has a length equal to that of its projection on the vertical, or in symbols

$$ds/dh = \pm 1. \qquad (10)$$

There remains, however, this distinction between s and h, that whereas h increases to a maximum at the vertex and then decreases, s increases all the time.

With the approximation (10) and the fact (4), the vertical dynamical equation (8) becomes

$$\ddot{h} = \pm n\,(\dot{h})^2 - g. \qquad (11)$$

It is seen that (11) is of the same form as it would be in still air, so that we may regard h as already given by the solution of (11) which has been obtained in naturalized form in § 4.3 above. When h is known, s follows immediately from (10).

And when s is regarded as independent of the relative velocities r_X, r_Y, then the dynamical equations (6) and (7) are linear in r_X, r_Y as (1) and (2) were not. Further, in (1) and (2) both components r_X, r_Y occur in both equations, whereas in (6) and (7) the components have an equation each. For these reasons the approximation (9) makes a great simplification. Of course, it cannot be valid at the top of the flight where $r_H = 0$, but the proportion of the erroneous part to the whole will diminish as greater heights are attained, that is to say, as the practical usefulness of the method increases. The approximation will also improve with larger speeds of ascent and descent, that is to say, with steel spheres larger than a pea.

Our task is now to solve either (6) or (7); either will do for they are of the same form. We must first change to velocities relative to the earth. Let v_X denote the wind's speed, and $\dot{x}$ the ball's speed, both relative to the earth, in the direction of x increasing; then

$$r_X = \dot{x} - v_X, \qquad (12)$$

so that (6) may be written

$$d^2x/dt^2 + n\dot{s}\,dx/dt = n\dot{s}v_X. \qquad (13)$$

We must now integrate (13) twice in order to find the horizontal displacement of the sphere. Since $n\dot{s}$ and $n\dot{s}v_X$ at the sphere may be regarded as known functions of time, therefore (13) becomes integrable* when multiplied by

$$e^{\int n\dot{s}dt}, \text{ which is equal to } e^{ns}, \qquad (14)$$

where the arbitrary constant, being a mere constant multiplier to (13), has been omitted.

A first integral of (13) is accordingly

$$\frac{dx}{dt} = \text{const.}\ e^{-ns} + e^{-ns}\int_0^t \frac{de^{ns}}{dt}\,v_X\,.\,dt. \qquad (15)$$

* FORSYTH, 'Differential Equations' (1903), Art. 14.

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The arbitrary constant is determined by the fact that at the beginning of the flight dx/dt = (muzzle-speed) $\times$(cosine of angle between barrel and the x axis)

$$= u\beta, \quad \text{for } \beta \text{ is this cosine.} \qquad (16)$$

Since we reckon t and s to be zero at the vertex, this value of $\dot{x}$ occurs when $t = -a$, where a is the time of ascent, and when $s = -l$, where l is the maximum height attained.

When the arbitrary constant has been removed from (15) in this way, we obtain

$$\frac{dx}{dt} = u\beta e^{-n(l+s)} + e^{-ns}\int_{-a}^{t} \frac{de^{ns}}{dt} v_X\, dt. \qquad (17)$$

Integrating once more we get the displacement of the sphere relative to the ground in the form

$$x = \text{const.} + u\beta\int_0^t e^{-n(l+s)}\, dt + \int_0^t e^{-ns}\int_{-a}^{t} \frac{de^{ns}}{dt} v_X\,.\, dt\,.\, dt. \qquad (18)$$

The arbitrary constant in (18) is determined by the convention that x is zero at the start, that is to say

$$x = 0 \quad \text{when} \quad t = -a \quad \text{and} \quad s = -l. \qquad (19)$$

Accordingly

$$x = u\beta\int_{-a}^{t} e^{-n(l+s)}\, dt + \int_{-a}^{t} e^{-ns}\int_{-a}^{t} \frac{de^{ns}}{dt} v_X\,.\, dt\,.\, dt. \qquad (20)$$

Now, the observer aims to make the returning ball hit the shelter which protects the gun, and if he does not quite succeed, he corrects the tilt to correspond to this condition, which in symbols is

$$x = 0 \quad \text{when} \quad t = b, \text{ the time of descent.} \qquad (21)$$

This condition converts (20) into the following " integral equation,"

$$0 = u\beta\int_{-a}^{b} e^{-n(l+s)}\, dt + \int_{-a}^{b} e^{-ns}\int_{-a}^{t} \frac{de^{ns}}{dt} v_X\,.\, dt\,.\, dt. \qquad (22)$$

The unknown speed $\dot{x}$ is now removed, and it remains to find v_X from β. The essence of the process is that we make shots going to various heights and so vary both limits of the integrals together with u and β.

But first suppose that we are shooting spheres, of a given n, to one height only; then the coefficient of β in (22) is a fixture; and if cosines of elevation β', β'' correspond relatively to wind distributions v_X', v_X'', then by substituting these values in (22) and adding the equations it is seen that a wind distribution $v_X' + v_X''$ would require to balance it an elevation having a cosine equal to $\beta' + \beta''$.

Or, more generally :—

For a fixed height and fixed terminal velocity, the sum of any number of wind distributions is balanced by the sum of the cosines of elevation which balance them individually. . (23)

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This convenient property depends on (22) being linear in β and in v_X, and the linearity is a consequence of the approximation (9).

It should be noted that the cosine of elevation is in practice almost indistinguishable from the tilt in radians from the vertical.

A Lamina of Wind.

As an elementary wind distribution we may imagine calm everywhere except for a lamina between h_i and $h_i + dh_i$ in which the velocity is uniform and equal to v_i. The advantage of such artificial elements is the ease with which any natural distribution may be built up from them by superposition. Let us find the result of this substitution on our fundamental equation (22).

First consider the inner integral, which changes only in crossing the lamina

$$\int_{-a}^{t} \frac{de^{ns}}{dt} v_X \, . \, dt \equiv \int_{-l}^{s} ne^{ns} \, . \, v_X \, ds = q, \text{ say, for short.} \qquad (24)$$

In rising to the lamina $v_x = 0$, and so $q = 0$. . . . . (25)

Above the lamina from $s = (h_i + dh_i - l)$ to $s = (l - h_i - dh_i)$, the value of q is due to the sphere having risen through the lamina, and is

$$q = ne^{n(h_i - l)} v_i \, . \, dh_i. \qquad (26)$$

In falling from the lamina, from $s = l - h_i$ to $s = l$, the second crossing makes

$$q = n \{ e^{n(h_i - l)} + e^{n(l - h_i)} \} v_i \, . \, dh_i. \qquad (27)$$

On inserting these in the outer integral in (22) we obtain

$$\int_{-a}^{b} e^{-ns} \int_{-a}^{t} \frac{de^{ns}}{dt} v_X \, dt \, dt \equiv \int_{-a}^{b} e^{-ns} \, . \, q \, . \, dt$$

$$= ne^{n(h_i - l)} v_i \, dh_i \int_{s = h_i - l}^{s = l - h_i} e^{-ns} \, dt + n \{ e^{n(h_i - l)} + e^{n(l - h_i)} \} v_i \, dh_i \int_{s = l - h_i}^{s = l} e^{-ns} \, dt. \qquad (28)$$

Inside the lamina q may be ignored, for it lies between its values outside.

Now a numerical table of $\displaystyle\int_{s=0}^{s=s} e^{-ns} \, dt = j$, say, for short,. . . . . (29)

expressed as a function of s is given below on page 360 in a naturalized form, so that (22) simplifies in this case to

$$0 = u \, . \, d\beta \, . \, e^{-nl} (j_{+l} - j_{-l}) + nv_i \, dh_i \left[e^{n(h_i - l)} \{ j_l - j_{h_i - l} \} + e^{n(l - h_i)} \{ j_l - j_{l - h_i} \} \right], \qquad (30)$$

in which $d\beta$ is written in place of β to correspond with dh_i.

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The portion in the square bracket depends, for a given sphere, only on l the maximum height attained, and on h_i the height of the lamina, so that we may denote the square bracket briefly by

$$f(l, h_i) \qquad (31)$$

where f is a function that is tabulated, in naturalized form, below.

The same function occurs in the first term, in the degenerate form $f(l, 0)$ so that (30) may be written

$$0 = u \, . \, d\beta \, . \, f(l, 0) + nv_i \, dh_i f(l, h_i). \qquad (32)$$

Then if, keeping l fixed, we superpose laminæ of wind one upon another, we have in view of the principle (23)

$$0 = u \, . \, \beta \, . \, f(l, 0) + \int_0^l f(l, h_i) \, nv_i \, dh_i, \qquad (33)$$

giving the tilt β due to any arbitrary distribution of wind. Thus, by the idea of laminæ, we have got rid of the double integral in (22) and have reached in (33) a "linear integral equation of the first kind" in something like its standard form.

Transformation to Natural Units.

To make our formulæ easily applicable to spheres of various sizes it is convenient now to bring in the "naturalized" quantities of § 4.2. To get these we need the terminal speed k. But the retardation due to the terminal speed is equal to the acceleration of gravity g; and the retardation due to the speed r has been defined to be nr^2.

Therefore $$nk^2 = g. \qquad (34)$$

Now, refer all velocities to k as unit, $\qquad$ (35)

and denote their new measures by the capital letters corresponding to the small letters.

Thus $$\mathrm{U} = \frac{u}{k} = u\sqrt{\frac{n}{g}}; \qquad \mathrm{V_X} = v_\mathrm{X}\sqrt{\frac{n}{g}}. \qquad (36)$$

Similarly, refer all times to k/g as unit, $\qquad$ (37)

putting for example $$\mathrm{T} = tg/k = t\sqrt{ng}. \qquad (38)$$

Also refer all lengths to k^2/g as unit, $\qquad$ (39)

so that $$\mathrm{S} = sg/k^2 = ns; \quad \mathrm{L} = nl; \quad \mathrm{H} = nh. \qquad (40)$$

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In this paper capital letters always stand for such "naturalized" measures. Some information concerning the numerical values of k will be found on page 351.

Our fundamental equation (22) then transforms into

$$0 = \mathrm{U}\beta \int_{-\mathrm{L}}^{\mathrm{L}} e^{-(\mathrm{L}-\mathrm{S})} \cdot d\mathrm{T} + \int_{\mathrm{T}_{-\mathrm{L}}}^{\mathrm{T}_\mathrm{L}} e^{-\mathrm{S}} \int_{\mathrm{T}_{-\mathrm{L}}}^{\mathrm{T}} \frac{de^{\mathrm{S}}}{d\mathrm{T}} \mathrm{V}_\mathrm{X}\, d\mathrm{T}\, .\, d\mathrm{T}, \quad . \quad . \quad . \quad . \quad (41)$$

where $\mathrm{T}_{-\mathrm{L}}$, T_L are the values of T at the beginning and end of the flight. Again j, the oft recurring function of s, transforms thus

$$j = \int_0^s e^{-ns}\, dt = \int_0^s e^{-ns} \frac{dt}{ds} \cdot ds = \int_0^{s=s} e^{-\mathrm{S}} \cdot \frac{d\mathrm{T}}{d\mathrm{S}} \sqrt{\frac{n}{g}} \cdot \frac{d\mathrm{S}}{n} = \frac{1}{\sqrt{ng}} \int_0^{\mathrm{S}=\mathrm{S}} e^{-\mathrm{S}}\, d\mathrm{T}.$$

Let us denote

$$\int_0^{\mathrm{S}} e^{-\mathrm{S}}\, d\mathrm{T} \text{ by the symbol J.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Then $j = \mathrm{J}/\sqrt{ng}$, as is otherwise obvious, because j is a time and $1/\sqrt{ng}$ is the "natural" unit of time.

It is convenient to indicate the value of S to which J corresponds by writing it as a suffix thus: J_H or J_L.

And then if we define, by analogy with the function f, a function F thus

$$\mathrm{F}(\mathrm{L}, \mathrm{H}) = e^{\mathrm{H}-\mathrm{L}} \{\mathrm{J}_\mathrm{L} - \mathrm{J}_{\mathrm{H}-\mathrm{L}}\} + e^{\mathrm{L}-\mathrm{H}} \{\mathrm{J}_\mathrm{L} - \mathrm{J}_{\mathrm{L}-\mathrm{H}}\}. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

it is found that the integral equation (33) transforms into

$$0 = \beta \, . \, \mathrm{U} \, . \, \mathrm{F}(\mathrm{L}, 0) + \int_0^{\mathrm{L}} \mathrm{F}(\mathrm{L}, \mathrm{H}) \, . \, \mathrm{V} \, . \, d\mathrm{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

To Tabulate J as a Function of s.

In the ascent, according to (8) of § 4.3,

$$e^{-\mathrm{S}} = \sec \mathrm{T},$$

and $$\int \sec \mathrm{T} \, . \, d\mathrm{T} + \text{const.} = \log \left\{\tan\left(\frac{\pi}{4} + \frac{\mathrm{T}}{2}\right)\right\} = \log \left\{\tan\left(\frac{\pi}{4} + \tfrac{1}{2} \sec^{-1} e^{-\mathrm{S}}\right)\right\}.$$

To make the integral vanish when $\mathrm{S} = 0$ the arbitrary constant must vanish.

Now since* $\log \left\{\tan\left(\tfrac{1}{2} \sec \cosh \psi + \frac{\pi}{4}\right)\right\} = \psi$ for any value of ψ. Therefore $\int_0 e^{-\mathrm{S}}\, d\mathrm{T} = \cosh^{-1} e^{-\mathrm{S}}$.

* JAHNKE & EMDE, 'Functionentafeln' (Teubner), p. 14.

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In the descent according to equation (11) of § 4.3,

$$e^{-S} = \frac{1}{\cosh T},$$

therefore

$$J + \text{const.} = \int \operatorname{sech} T \,.\, dT = \sin^{-1} \tanh T,$$

$$= \cos^{-1} e^{-S}.$$

Since the last form vanishes when $S = 0$ the arbitrary constant vanishes. On collecting results we find :—

Ascent.	Descent.
$T = \sec^{-1} e^{-S}$,	$J = \sec^{-1} e^{S}$,
$J = \cosh^{-1} e^{-S}$,	$T = \cosh^{-1} e^{S}$.

It is seen that there is a curious relationship between T and J such that T at any level on one side of the vertex is equal to [minus] J at the same level on the other side. The signs are left ambiguous by the formulæ in $\cosh^{-1}$, $\cos^{-1}$, but the integrals show that J, like T, increases as the ball moves on, so that we must insert the minus in the preceding statement.

It follows that

$$T_L - T_{-L} = -J_{-L} + J_L,$$

so that we can obtain the whole time of absence from the table giving J as a function of S. This curious relation was first noticed in the arithmetic and afterwards proved.

From these formulæ the following table has been computed.

TABLE III.

S.	J.	S.	J.
—3·0	—3·6925	0·0	0·0000
—2·8	—3·4922	0·2	0·6116
—2·6	—3·2918	0·4	0·8362
—2·4	—3·0911	0·6	0·9898
—2·2	—2·8901	0·8	1·1048
—2·0	—2·6885	1·0	1·1940
—1·8	—2·4862	1·2	1·2649
—1·6	—2·2828	1·4	1·3216
—1·4	—2·0776	1·6	1·3675
—1·2	—1·8697	1·8	1·4047
—1·0	—1·6574	2·0	1·4350
—0·8	—1·4383	2·2	1·4598
—0·6	—1·2076	2·4	1·4799
—0·4	—0·9551	2·6	1·4964
—0·2	—0·6539	2·8	1·5100
0·0	0·0000	3·0	1·5209

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§ 4.5. *Approximate General Solution of the Integral Equation.*

Our starting point is now equation (44) of § 4.4 which reads

$$0 = \beta . U . F(L, 0) + \int_0^L F(L, H) . V . dH. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let there be a series of increasing heights H_0, H_1, H_2, H_3, etc. Let spheres, all alike, be shot up to the alternate heights H_2, H_4, H_6, etc., let U_2, U_4, U_6, etc., be the corresponding muzzle-speeds, and let the balancing tilts be measured by β_2, β_4, β_6, etc., where β is the cosine of the angle between the barrel and a horizontal line fixed in, say, the east or north direction. Let V_1, V_3, V_5 be wind-speeds of the intermediate layers. Next, assume that the layers are so thin that for any one of them we may replace the integral by the range of H multiplied by the value of the integrand at the intermediate height, so that taking for example the layer between H_2 and H_4 we put

$$\int_{H_2}^{H_4} F(L, H) . V . dH = F(L, H_3) . V_3 . (H_4 - H_2). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then, for a shot to H_2, equations (1) and (2) give

$$0 = \beta_2 . U_2 . F(L_2, 0) + F(L_2, H_1) . V_1 . (H_2 - H_0), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is a simple algebraic equation giving the naturalized wind-speed V_1 of the first layer in terms of tabulated quantities and of the observed β_2.

For a shot to the next greater height, H_4, we have from (1) and (2), since the integral splits into two parts in series

$$0 = \beta_4 . U_4 . F(L_4, 0) + (H_2 - H_0) . V_1 . F(L_4, H_1) + (H_4 - H_2) . V_3 . F(L_4, H_3). \quad . \quad (4)$$

This is a linear algebraic equation between V_3, V_1 and β_4 with known coefficients, and we may substitute in it the value of V_1 from (3) and thus obtain V_3.

Proceeding to the shot reaching H_6, we obtain in a similar way a linear algebraic equation between V_1, V_3, V_5, and of these all but V_5 are known. Hence V_5 is determined. And so on without limit.

This transformation to algebraic equations is due to Volterra.*

Now, in order to express the solution in numbers, let us make the series H_0, H_1, H_2, H_3 have a common difference of 0·2 so that shots rise to naturalized heights of 0·4, 0·8, 1·2, 1·6, and so on. Then a table of the kernel F (L, H) has been prepared by the aid of equation (43) of § 4.4 and the preceding table of J. It is further necessary to tabulate U . F (L, 0). These two tables are given together below.

* *See* 'An Introduction to the Study of Integral Equations,' Art. 7, by MAXIME BÔCHER, Cambridge University Press.

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TABLE IV.

L.	U.F (L, 0).	The Kernel F (L, H).						
		H = 0·0.	H = 0·2.	H = 0·6.	H = 1·0.	H = 1·4.	H = 1·8.	H = 2·2.
0·0	0·0000	0·0000						
0·4	1·3305	1·2017	1·4943					
0·8	2·2719	1·1427	1·4786	2·0423				
1·2	2·9889	0·9441	1·2678	1·8582	2·3689			
1·6	3·5752	0·7370	1·0357	1·5844	2·1015	2·5780		
2·0	4·0859	0·5581	0·8315	1·3261	1·7927	2·2615	2·7160	
2·4	4·552	0·4147	0·6656	1·1105	1·5192	1·9313	2·3679	2·8076

(NOTE.—The underlined final digits are uncertain.)

With the aid of this table it was possible to write down six algebraic equations which together represent the integral equation, in accordance with the approximation (2) above. Thus

$$\left.\begin{aligned} 0 &= \beta_{0\cdot4}.3\cdot3262+V_{0\cdot2}.1\cdot4943,\\ 0 &= \beta_{0\cdot8}.5\cdot6798+V_{0\cdot2}.1\cdot4786+V_{0\cdot6}.2\cdot0423,\\ 0 &= \beta_{1\cdot2}.7\cdot4722+V_{0\cdot2}\,1\cdot2678+V_{0\cdot6}\,1\cdot8582+V_{1\cdot0}\,2\cdot3689,\end{aligned}\right\} \quad (5)$$

and three others built on the same plan. The suffixes are here the values of L. To form the equation containing, say, $\beta_{1\cdot2}$ look at the row beginning with 1·2 in the preceding table. The coefficient of $\beta_{1\cdot2}$ is the next number in this row divided by the common difference, 0·4 of L. Skipping over the column headed H = 0·0, the coefficients of $V_{0\cdot2}$, $V_{0\cdot6}$, etc., are simply the successive numbers in the same row.

These six equations give immediately the tilts required to balance a known distribution of naturalized wind. Our problem, however, is the converse: given the tilt to find the wind. So the equations (5) were solved for the velocities by the aid of a multiplying machine, with the following result :—

$$\left.\begin{aligned} V_{0\cdot2} &= -2\cdot226\beta_{0\cdot4},\\ V_{0\cdot6} &= -2\cdot781\beta_{0\cdot8}+1\cdot612\beta_{0\cdot4},\\ V_{1\cdot0} &= -3\cdot154\beta_{1\cdot2}+2\cdot182\beta_{0\cdot8}-0\cdot073\beta_{0\cdot4},\\ V_{1\cdot4} &= -3\cdot467\beta_{1\cdot6}+2\cdot571\beta_{1\cdot2}-0\cdot069\beta_{0\cdot8}-0\cdot037\beta_{0\cdot4},\\ V_{1\cdot8} &= -3\cdot761\beta_{2\cdot0}+2\cdot887\beta_{1\cdot6}-0\cdot059\beta_{1\cdot2}-0\cdot025\beta_{0\cdot8}-0\cdot027\beta_{0\cdot4},\\ V_{2\cdot2} &= -4\cdot053\beta_{2\cdot4}+3\cdot172\beta_{2\cdot0}-0\cdot050\beta_{1\cdot6}-0\cdot012\beta_{1\cdot2}-0\cdot012\beta_{0\cdot8}-0\cdot023\beta_{0\cdot4}\end{aligned}\right\} \quad (6)$$

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In order to apply these equations to observation, we must know the "natural units" for the projectile used. Some information will be found on page 351.

It is seen that the first two terms in these equations are much more important than the following terms, that is to say, the naturalized wind V at any level can be expressed nearly in terms of β and $d\beta/d\mathrm{L}$ at the same level. This unexpected fact simplifies the observer's work.

Finally, to test the error due to the number of layers not being infinite, we may repeat the process with layers of a different thickness. Thus, for layers three times as thick it is found in a similar way, that

$$V_{0\cdot6} = -1\cdot340\beta_{1\cdot2},$$
$$V_{1\cdot8} = -1\cdot602\beta_{2\cdot4} + 0\cdot629\beta_{1\cdot2}.$$

To contrast the two solutions, take the simplest case, in which all the β are equal. Then :—

By layers of 1·2	$V_{0\cdot6} = -1\cdot340\beta$	$V_{1\cdot8} = -0\cdot973\beta$
By layers of 0·4	$V_{0\cdot6} = -1\cdot169\beta$	$V_{1\cdot8} = -0\cdot985\beta$
Difference . . .	$-0\cdot171\beta$	$+0\cdot012\beta$

As for both thicknesses the differences are centred, and centred in just the same way, and as the results given by the two formulæ do not differ greatly, we may expect that the error is proportional to the square of the thickness of the layer ;* that is to say, the errors are as 9 : 1 and the true result may be found by extrapolation thus

$$V_{0\cdot6} = \left(-1\cdot169 + \tfrac{1}{8}\,0\cdot171\right)\beta = -1\cdot148\beta,$$
$$V_{1\cdot8} = \left(-0\cdot985 - \tfrac{1}{8}\,0\cdot012\right)\beta = -0\cdot986\beta.$$

It is thought that this way of approaching the limit, when the layers are infinitely thin, is more convenient than FREDHOLM'S series of multiple integrated determinants.†

In this example the errors due to finite differences of 0·4 in L are seen to be 2 per cent. and 0·1 per cent. In view of the gustiness of the wind these scarcely matter, and errors due to finite differences are usually ignored in the sequel. However, as a check, two solutions free from this error will now be given for special cases.

§ 4.6. *Analytical Solutions for Special Cases.*

These have been derived from equation (22) of § 4.4. For brevity the results are stated without proof.

* L. F. RICHARDSON. 'Phil. Trans.,' A, Vol. 210, p. 310 (1910).

† Described by BÔCHER, 'Introduction to the Study of Integral Equations,' Cambridge Press, p. 32.

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§ 4.6.1. *Wind Speed and Direction both Independent of Height.*

Let v be the speed and let β be the cosine of the angle between the barrel and a horizontal line in the direction of the wind. Then

$$v = -\frac{\beta u}{e^{nl}-1} \quad \ldots \quad (1)$$

which in naturalized quantities reads

$$\mathrm{V} = -\frac{\beta \mathrm{U}}{e^{\mathrm{L}}-1} \quad \ldots \quad (2)$$

The following table shows how the tilt increases with the height attained. Contrast the distribution, discussed on page **363**, in which the tilt was independent of height and the wind speed decreased aloft.

TABLE V.

L	$-\beta = \mathrm{V} \times$ (the number printed below).	
	By Infinitesimals. (Eqn. (2)).	By the General Approximate Solution in steps of 0·4.
0·4	0·444	0·449
0·8	0·616	0·620
1·2	0·732	0·735
1·6	0·815	0·817
2·0	0·872	0·874
2·4	0·913	0·914

Here again the error due to finite differences is insignificant.

§ 4.6.2. *The Range in Still Air and the Correction for the Point of Fall.*

By the range is meant the horizontal distance of the point of fall from the gun. By § **3** this is

$$u\beta(t'-c) \quad \ldots \quad (1)$$

where c is such that a uniform wind v is balanced by β when

$$v = -u\beta . \frac{t'-c}{c} . \quad \ldots \quad (2)$$

On comparing this with the formula just given for a uniform wind, it is seen that c is given by

$$\frac{t'-c}{c} = \frac{1}{e^{nl}-1} = \frac{1}{e^{\mathrm{L}}-1} . \quad \ldots \quad (3)$$

On eliminating c between (1) and (3) it is found that

$$\text{range in still air} = u . \beta . t' . e^{-\mathrm{L}} . \quad \ldots \quad (4)$$

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That is to say, the sphere travels horizontally e^{-L} of the distance it would have gone in the same time in the absence of horizontal retardation. This is probably well-known.

Formula (4) is used to find the amount by which β must be adjusted to allow for the horizontal distance by which the returning ball misses its target, namely, the roof over the gun.

$$\text{The correction to add to } \beta = \pm \frac{\text{naturalized horizontal distance}}{\mathrm{UT}'e^{-\mathrm{L}}} \quad . \quad (5)$$

The factor is tabulated below :—

TABLE VI.

L.	T′.	U.	$1/(\mathrm{UT}'e^{-\mathrm{L}})$.
(0·2	1·266	0·71	1·36)
0·4	1·791	1·107	0·752
0·8	2·543	1·988	0·441
1·2	3·135	3·166	0·335
1·6	3·650	4·851	0·280
2·0	4·124	7·321	0·245
2·4	4·571	10·978	0·220

§ 4.6.3. *Uniform Wind above Calm.*

This distribution has been examined for comparison with that sometimes observed on cold mornings before sunrise.

The sphere is supposed to ascend to a naturalized height L, of which the upper part, Z, is in the wind, so that the naturalized thickness of the calm is L-Z. The deduction which, being rather long, is here omitted, begins from the integral equation (22) of § 4.4 and ends in

$$\mathrm{V} = \frac{-\beta \,.\, \mathrm{U}e^{-\mathrm{L}}\left(\mathrm{J_L}-\mathrm{J_{-L}}\right)}{\mathrm{T_Z}-\mathrm{T_{-Z}}+e^{-\mathrm{Z}}\mathrm{J_{-Z}}-e^{\mathrm{Z}}\mathrm{J_Z}+\mathrm{J_L}\left(e^{\mathrm{Z}}-e^{-\mathrm{Z}}\right)}.$$

The coefficient of β for any observed values of Z and L may be computed from the tables given above. For instance, when L = 1·4, Z = 0·2, the formula reduces to V = −6·4β. It will be shown in § 5 that this formula requires a correction dependent on V^3 at the vertex and intended to compensate for the neglect of $\mathrm{R_X}^2$ in $\sqrt{\mathrm{R_H}^2+\mathrm{R_X}^2}$ near the vertex. The importance of the correction decreases as Z increases and V diminishes.

§ 4.7. *Illustration.*

To illustrate the application of the foregoing theory an account will now be given of an *Observation made at Benson* (Long. 1° 6′ W. ; Lat. 51° 37′ N., on June 18, 1920, between 4 h. 0 m. and 4 h. 26 m. G.M.T.). The readings are plotted on the accompanying diagram.* The vertical co-ordinate of the diagram is marked off in three scales, one of ordinary height, one of naturalized height and one of the observed time of absence. The relations between the three scales were computed from the formulæ given in § 4.3 above on the assumption that the terminal speed of descent of the balls was 59·9 metres per second. The horizontal co-ordinate is the cosine which we have called β of the angle between the barrel and the horizontal axis. It has first been corrected for missing the bulls-

* Fig. 3 on page 366.

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eye. As the displacements recorded as positive to the right are those of the butt instead of those of the muzzle, it follows that β is negative in the direction of the displacement. Crosses mean that the butt was displaced to the east, circles or squares mean that it was displaced to the north. The values are read from the smoothed lines at the heights L = 2·0, L = 1·6. Then by the approximate general solution of page 362 above, we have $V_{1\cdot8} = -3\cdot761\beta_{2\cdot0} + 2\cdot887\beta_{1\cdot6}$ plus terms in $\beta_{1\cdot2}$, $\beta_{0\cdot8}$, $B_{0\cdot4}$ with small coefficients. These terms are negligible unless the lower wind is unusually strong. On the present occasion preliminary shots with a "Daisy" Air-gun had shown that the mean wind speed up to 120 metres above the gun was only about 3 m/s., so that it was justifiable to neglect the terms in question.

Now putting in β as read from the chart and then multiplying $V_{1\cdot8}$ by the terminal speed of the ball, which is taken to be 59·9 m/s. we find for the wind components, 2·5 m/s. from W, 5·7 m/s. from S at the level L = 1·8 which is seen from the diagram to correspond to 660 metres above the gun.

An interesting point about this occasion is that the two lowest shots which were the first made, were fired into a cloudless sky; and that then a low cloud blew over at the level of the tree tops and the subsequent observations were made through it.

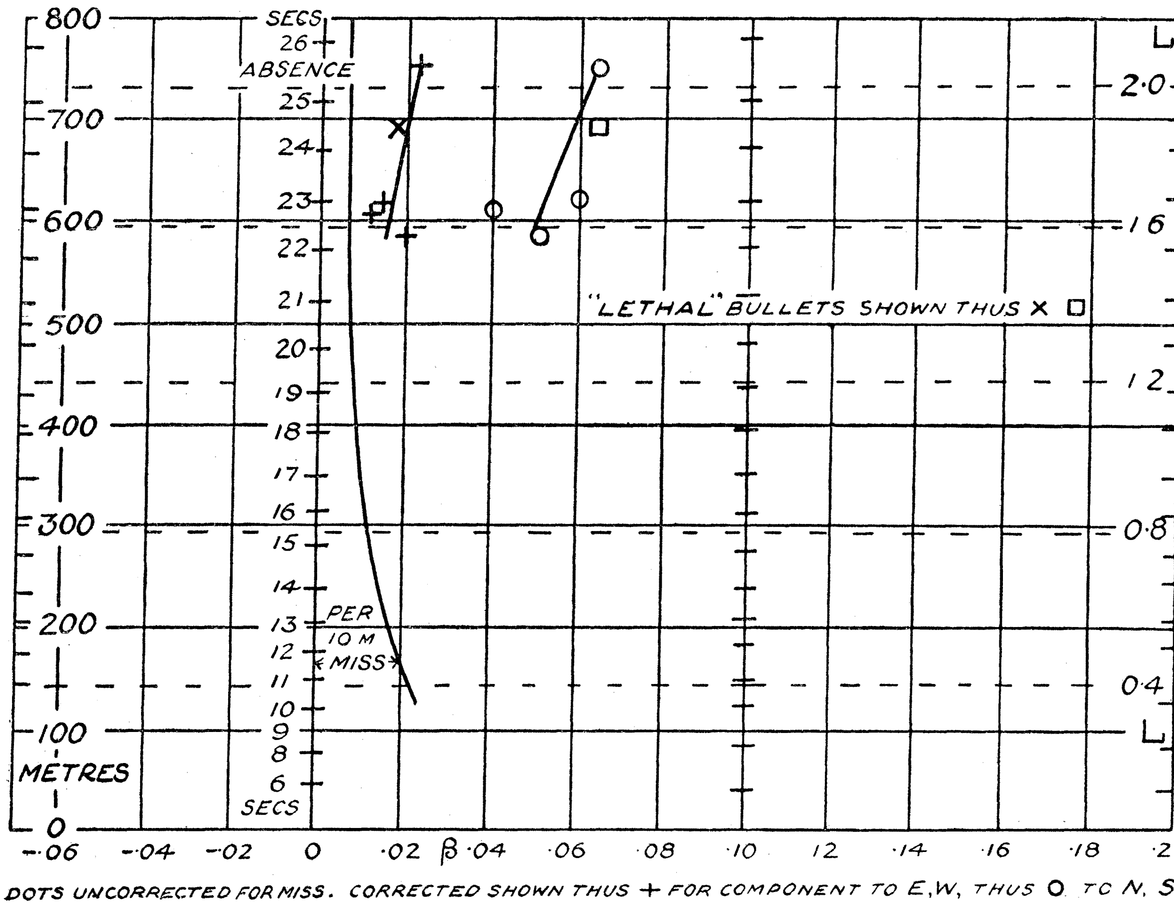

Fig. 3. Observer's chart for balls having terminal speed of 59·9 m/s. Wind above cloud. BENSON, 1920, June 18d. 4h. (sunrise).

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§ 5. Improvements near the Vertex.

§ 5.1. *Introduction.*

Some of the most interesting observations yet obtained by this new method are those like the one shown in figure (4) in which the balancing tilt is zero for times of absence less than a certain amount, and increases very rapidly as the times increase beyond this amount. This distribution obviously indicates a wind above separated sharply from a calm below. But the strength of the wind just above the discontinuity is to be measured by the motion of the sphere near its vertex, and so must remain an open question until we have improved the theory of this part of the motion. For in order to have a linear relation between tilt and wind-speed we have hitherto assumed (*see* § 4) that r_H is a sufficient approximation to $\sqrt{(r_X{}^2+r_Y{}^2+r_H{}^2)}$. This assumption fails entirely near the vertex where r_H vanishes.

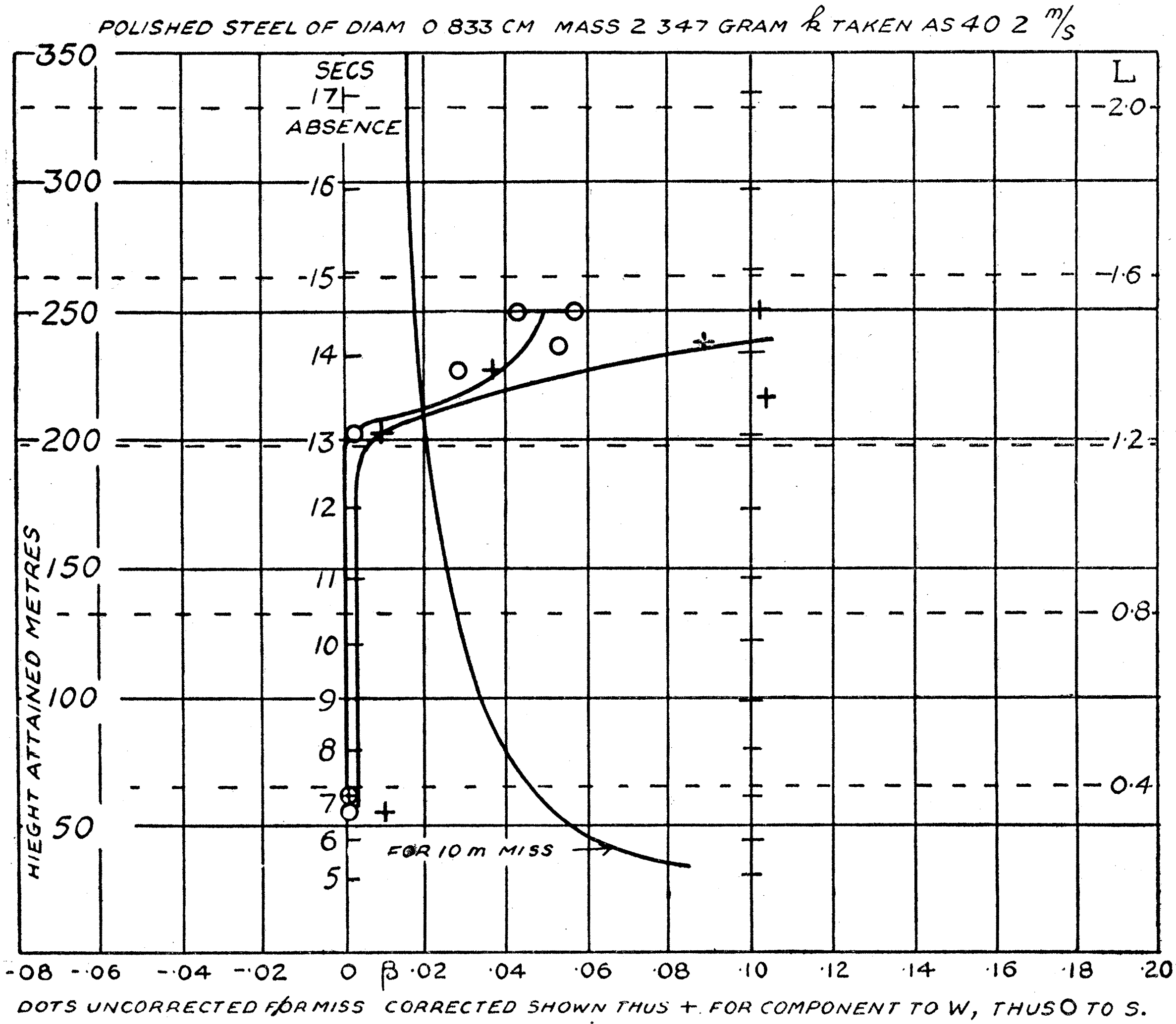

Fig. 4. Observer's chart for balls having a terminal speed of 40·2 m/s. Wind above calm. Benson, 1919, Nov. 27d. 7h.

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§ 5.2. *Theory when the Sphere Penetrates a Little into a Wind Blowing over Calm.*

§ 5.2.1. As an apology for various approximations that have to be made, it should be stated that the integration of the equations of motion of a sphere moving in a calm and experiencing a resistance proportional to the square of its velocity is a problem two centuries old, and, according to CRANZ and BECKER,* although it is possible to express the relations between the path, the speed, and the direction of motion in a finite form, yet the height, the horizontal displacement and the time cannot be found without quadratures.

Let our object here be to make a theory which shall supplement that of § 4.6.3 by (i) tending to exactitude when the sphere only just enters the wind; (ii) becoming only gradually erroneous as the penetration and the wind-speed increase, so as to reach out towards the region in which the theory of § 4.6.3 is valid, and in particular so as to be suitable for the reduction of the observation shown in fig. (4); (iii) by not involving quadratures.

Later in § 5.3 the approximations will be tested by quadratures applied to a particular instance. The objects stated above have been attained as follows:—The sphere is supposed to move up to the vertex through a calm according to the simple theory of § 4 in which $\sqrt{\{1+(R_X/R_H)^2\}}$ is put equal to unity, and to move down again from the vertex according to the same theory, keeping in calm air all the way. But at the vertex a discontinuity $\dot{X}_2-\dot{X}_1$ in the horizontal velocity is introduced, equal in amount to the change of horizontal velocity $R_{X2}-R_{X1}$ which the sphere would experience, according to a more exact theory, in moving through a thin layer of uniform wind extending from the vertex to a naturalized depth Z below. It would, no doubt, be better to introduce a discontinuity of position at the vertex as well as one of velocity, but that apparently cannot be done without quadratures and its omission does not seem to matter much. In the limit $Z \rightarrow 0$ it matters not at all.

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§ 5.2.2. *Relation between Velocities at the Interface.*

The principle of the addition of velocities gives us: (speed of sphere relative to air)+(speed of air relative to earth)=(speed of sphere relative to earth) which in symbols reads

$$R_X+V_X = \dot{X}. \qquad (1)$$

Let suffix 1 denote quantities at the point where the sphere enters the wind, suffix 2 those where it leaves.

Then

$$R_{X1}+V_X = \dot{X}_1, \qquad (2)$$

$$R_{X2}+V_X = \dot{X}_2. \qquad (3)$$

By subtraction

$$R_{X2}-R_{X1} = \dot{X}_2-\dot{X}_1, \qquad (4)$$

* 'Handbook on Ballistics,' § 17, § 20. H.M. Stationery Office, 1921.

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the second member of this is the change in the speed relative to earth, which we choose to regard as if it were concentrated as a discontinuity at the vertex. The first member can be found in terms of Z, and of R_0 the value of R_X at the vertex; as will be shown.

Again adding (2) and (3) and dividing by 2,

$$V_X = \tfrac{1}{2}(\dot{X}_1+\dot{X}_2) - \tfrac{1}{2}(R_{X1}+R_{X2}), \quad (5)$$

which is the equation which will ultimately give us V_x when in its second member the first term has been found from the theory of the calm and the second from the better theory of the wind.

§ 5.2.3. *Nearly Vertical Motion in Calm, with Discontinuity at Vertex.*

As the air is calm

$$V_X = 0 \quad (1)$$

and so from $R_X+V_X = \dot{X}$ it follows that

$$dR_X/dT = d\dot{X}/dT. \quad (2)$$

The horizontal dynamical equation, taken from (1) of § 4.4 and expressed in natural units, then becomes

$$dR_X/dT = -R_X|\sqrt{(R_X^2+R_{II}^2)}|. \quad (3)$$

That is

$$dR_X/dS = -R_X. \quad (4)$$

On integrating (4) and introducing (1) and (2) it follows that

$$\dot{X} = \text{const. } e^{-S}. \quad (5)$$

In the *ascent* starting at $S = -L$ and $\dot{X} = \beta U$ we have from (5) on inserting a value of the constant to give these initial conditions

$$\dot{X} = \beta U e^{-L-S}. \quad (6)$$

At the end of the ascent, just before the impulse, let

$$\dot{X} = \dot{X}_1, \quad (7)$$

then from (6) as $S = 0$ at the vertex

$$\dot{X}_1 = \beta U e^{-L}. \quad (8)$$

Just after the impulse, at the beginning of the descent, let

$$\dot{X} = \dot{X}_2. \quad (9)$$

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Then during the descent, by inserting a constant in (5) so as to satisfy (9) when $S = 0$,

$$\dot{X} = \dot{X}_2 e^{-S}. \qquad (10)$$

The whole naturalized horizontal displacement in the ascent is found by integrating (6) and comes to

$$\beta U e^{-L} \int_{S=-L}^{S=0} e^{-S} \, . \, dT = -\beta U e^{-L} J_{-L} \qquad (11)$$

where J_{-L} has already been tabulated on page 360.

Similarly, the whole naturalized displacement in the descent is found from (10) and comes to

$$\dot{X}_2 \int_{S=0}^{S=L} e^{-S} \, dT = \dot{X}_2 J_L \qquad (12)$$

As the tilt is adjusted so as to make the returning ball hit the gun, the sum of these two displacements must be zero; that is to say

$$\dot{X}_2 = \beta U e^{-L} J_{-L}/J_L. \qquad (13)$$

The jump in naturalized speed at the vertex is $\dot{X}_2 - \dot{X}_1$. But from (8) and (13)

$$\dot{X}_2 - \dot{X}_1 = -\beta U e^{-L} (1 - J_{-L}/J_L) \qquad (14)$$

The coefficient of β in (14) is a pure number, a function of the naturalized height attained. It has been computed from the previously given tables with the results shown in the second column below.

TABLE VII.

L.	$\phi(L) \equiv Ue^{-L}(1 - J_{-}\ /J_L)$.	$\mu(L) \equiv Ue^{-L}\frac{1}{2}(1 + J_{-L}/J_L)$.
0·0	0·00	—0·00
0·4	1·59	—0·05
0·8	2·06	—0·13
1·2	2·36	—0·23
(1·4)	(2·48)	—(0·28)
1·6	2·61	—0·33
2·0	2·84	—0·43
2·4	3·07	—0·54

This table, combined with equation (14), gives a satisfactory measure of the tilt required to balance a horizontal impulse at the vertex.

We also require $\frac{1}{2}(\dot{X}_1 + \dot{X}_2)$. This is found from (8) and (13) to be

$$\tfrac{1}{2}(\dot{X}_1 + \dot{X}_2) = \beta U e^{-L} \tfrac{1}{2} (1 + J_{-L}/J_L) \qquad (15)$$

The coefficient of β is set out in the third column of the accompanying table. It is seen to be small, that is to say, in the wind layer the sphere is moving slowly horizontally relative to the gun.

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For brevity the two tabulated functions are denoted by ϕ (L) and μ (L) as shown by the headings of their columns in the table.

Now, inserting (14) in (4) of the preceding § 5.2.2, the discontinuity is

$$R_{X_2} - R_{X_1} = -\beta\phi(L) \quad . \quad . \quad . \quad (16)$$

And inserting (15) in (5) of the preceding § 5.2.2, the wind-speed is

$$V_X = \beta \cdot \mu(L) - \tfrac{1}{2}(R_{X_1} + R_{X_2}). \quad . \quad . \quad . \quad (17)$$

The relations of R_{X1}, R_0, R_{X2} remain to be investigated by a better theory in the wind, to which we now pass on.

§ 5.2.4. *Motion in Uniform Wind; Approximation which becomes Exact at Vertex.*

Because the wind-speed is assumed to be the same all along this part of the trajectory so as in § 5.2.3,

$$dR_X/dT = d\dot{X}/dT. \quad . \quad . \quad . \quad (1)$$

The equations of motion, taken from § 4.4 are in " natural " units

$$dR_X/dT = -R_X|\sqrt{(R_X^2 + R_H^2)}|, \quad . \quad . \quad . \quad (2)$$

$$dR_H/dT = -R_H|\sqrt{(R_X^2 + R_H^2)}| - 1. \quad . \quad . \quad . \quad (3)$$

We are concerned with the region in which the vertical motion goes on almost as if *in vacuo*. On examining (3) this is seen to be the region in which

$$R_H|\sqrt{(R_X^2 + R_H^2)}| \text{ is much less than unity.} \quad . \quad . \quad . \quad (4)$$

So let us take simply

$$dR_H/dT = -1 \quad . \quad . \quad . \quad (5)$$

as describing the vertical motion.

On integrating this

$$R_H = -T \quad . \quad . \quad . \quad (6)$$

since R_H and T vanish together.

If, as elsewhere, we use the symbol Z to mean naturalized depth below the vertex then

$$Z = -\int_0^T R_H \cdot dT = \tfrac{1}{2}T^2. \quad . \quad . \quad . \quad (7)$$

Now, in the horizontal equation (2) let us divide through by R_X^2 obtaining*

$$-\frac{1}{R_X^2}\frac{dR_X}{dT} = |\sqrt{\{1 + (R_H/R_X)^2\}}|, \quad . \quad . \quad . \quad (8)$$

and then in view of (6) put for $(R_H/R_X)^2$ under the square root the approximate value

* Provided that R_X is positive. Let us suppose this to be so. If R_X were negative many signs in the following equations would have to be changed.

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$(T/R_0)^2$ where R_0 is the value of R_X at the vertex, near the middle of the arc under investigation. The horizontal dynamical equation thus becomes

$$-dR_X/R_X^2 = |\sqrt{\{1+(T/R_0)^2\}}|\,dT \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in which the variables are separated. On integrating and determining the arbitrary constant by

$$R_X = R_0 \quad \text{when} \quad T = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

it is found that

$$R_X = \frac{R_0}{1+R_0^2\Omega(\alpha)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where Ω is the functional symbol defined by

$$\Omega(\alpha) = \tfrac{1}{2}\alpha\sqrt{1+\alpha^2}+\tfrac{1}{2}\log(\alpha+\sqrt{1+\alpha^2}). \quad . \quad . \quad . \quad . \quad (12)$$

and

$$\alpha = T/R_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For our purpose α must be expressed as a function of Z.

This is done by (7) giving

$\alpha = \sqrt{2Z}/R_0$ with the added convention that the sign of α is that of T. . (14)

Thus, the sphere enters the wind layer at $\alpha = -|\sqrt{2Z}/R_0|$ and with $R_X = R_{X1}$; and leaves the wind again with

$$\alpha = +|\sqrt{2Z}/R_0| \text{ and with } R_X = R_{X2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The function Ω is such that

$$\Omega(\alpha) = -\Omega(-\alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It follows from these principles, after some algebraic simplifications, that

$$R_{X2}-R_{X1} = \frac{-2R_0^3\Omega_2}{1-R_0^4\Omega_2^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where Ω_2 is the value of Ω when

$$\alpha = +|\sqrt{2Z}/R_0|. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Similarly, it is found that

$$\tfrac{1}{2}(R_{X2}+R_{X1}) = \frac{R_0}{1-R_0^4\Omega_2^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Equations (17) and (19) give the desired connection with the theory in the calm. Thus (17) with (16) of the preceding § 5.2.3 gives

$$\beta\,.\,\phi(L) = 2R_0^3\Omega_2/(1-R_0^4\Omega_2^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

When β, L and Z have been observed, this equation yields R_0.

Next (19) with (17) of the preceding § 5.2.3 gives

$$V_X = \beta\mu(L) - R_0/(1-R_0^4\Omega_2^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

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and so our search for V_X is ended. There is really no need to go *via* R_0; all one requires is $R_0/(1-R_0^4\Omega_2^2)$ expressed as a function of $2R_0^3\Omega_2/(1-R_0^4\Omega_2^2)$. This relation is shown as a graph for Z = 0·1, 0·2, by plotting the numbers in the adjacent table VIII.

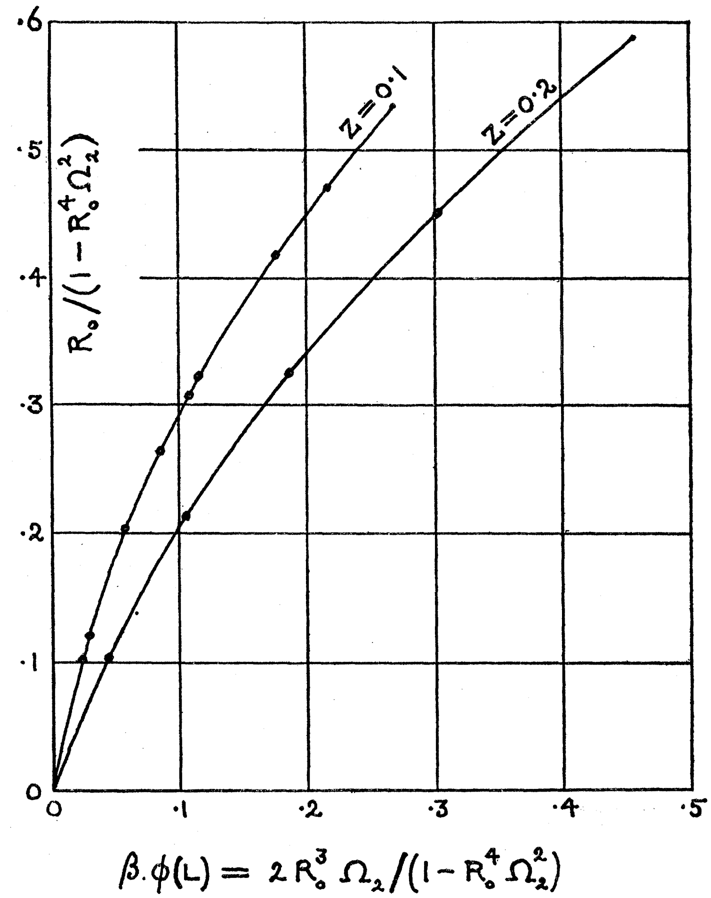

Fig. 5.

The functions involving Ω are tabulated below for values of the variables which are likely to be employed in reducing observations.

TABLE VIII.

R_0.	Z = 0·1.		Z = 0·2.	
	$\frac{R_0}{1-R_0^4\Omega_2^2}$.	$\frac{2R_0^3\Omega_2}{1-R_0^4\Omega_2^2}$.	$\frac{R_0}{1-R_0^4\Omega_2^2}$.	$\frac{2R_0^3\Omega_2}{1-R_0^4\Omega_2^2}$.
0·1	0·101	0·023	0·104	0·045
0·2	0·203	0·057	0·213	0·105
0·3	0·309	0·108	0·327	0·188
0·4	0·419	0·177	0·451	0·302
0·5	0·534	0·267	0·588	0·456

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The author regrets that this piece of theory is so involved, but his numerous attempts to find something reliable yet simpler have all ended in the waste-paper basket. For it is not enough that the theory should become exact as $Z \rightarrow 0$, because gustiness and personal error combine to hinder the application of theories in which the sphere "only just" does something. By contrast the theory presented above has sufficient range in Z and in V to be useful, as will be shown in § 5.3 by comparing it with a more exactly computed specimen.

The practical application is made easy by a table derived from (20) and (21) and published by the Meteorological Office in Professional Note No. 34.

Illustration of § 5.2.—*Observation of November* 27, 1919 (7 *h.*) *at Benson.*—From the observer's chart shown on p. 367, if we select for the vertex the naturalized height $Z = 0 \cdot 1$ above where the wind begins, then $L = 1 \cdot 3$ and resultant $\beta = 0 \cdot 05$.

Now, from the table on p. 370

$$\phi(L) = 2 \cdot 42,$$

and so by equation (20)

$$2R_0^3\Omega_2/(1-R_0^4\Omega_2^2) = 2 \cdot 42 \times 0 \cdot 05 = 0 \cdot 121.$$

Now turning to the graph of the two functions involving Ω, we take $0 \cdot 121$ as abscissa and read the ordinate of the curve marked $Z = 0 \cdot 1$. This ordinate is $0 \cdot 33$ and is ready for insertion in equation (21). Next by the table on p. 370

$$\mu(L) = -0 \cdot 255.$$

And so equation (21) gives

$$V_X = -0 \cdot 05 \times 0 \cdot 255 - 0 \cdot 33 = -0 \cdot 34.$$

The terminal speed k, in which, as unit, V_X is measured, is equal to $40 \cdot 2$ m/s. for these spheres, so it follows that the mean wind just above the calm was

$$0 \cdot 34 \times 40 \cdot 2 \text{ m/s.} = 13 \cdot 7 \text{ metres per second.}$$

This layer, $Z = 0 \cdot 1$, is 16 metres thick. If, instead, we find the mean wind in the first 33 metres ($Z = 0 \cdot 2$) above the calm, the result by the same process is

$$17 \cdot 0 \text{ metres per second.}$$

It is remarkable that such high speeds should exist so close to the calm.

§ 5.3. *A Test Case.*

Except in this section § 5.3 all the theory in this memoir is designed to enable the observer to compute the winds in, say, 15 minutes of time. To attain this speed, various approximations have been necessary. In this section, however, a calculation of a

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different type, more accurate and much more elaborate, is applied to an isolated example in order to see whether the approximations made in the rest of the memoir are justified. The distribution of wind selected for this test is likely to show up some of the errors in an extreme degree.

The calm is assumed to extend from the ground to a naturalized height of C. The velocity in the wind layer is taken to be independent of height. The sphere is assumed to penetrate the wind to a naturalized height of $Z = 0\cdot2$ above the interface. The problem is then to find the balancing tilt and the wind-speed. The solution consists of two parts; one in the wind, one in the calm. They have to be fitted together at the interface.

The Motion in the Wind.—Because the wind is independent of height we can reduce it to a calm by giving a suitable translational velocity to our axes of reference. The trajectory in this fictitious calm is then far from vertical, so that it is necessary to obtain the required integrals of the dynamical equations, without neglecting any terms. This is impossible without quadratures. There are, however, some analytic partway stages. The more useful of these, taken from CRANZ and BECKER* § 20, and translated into present notation, read as follows:—Let R_0 be the value of the naturalized speed relative to air at the vertex. Then

$$R_X = R_0 e^{-S}. \qquad (1)$$

Also

$$R_X^2 = \frac{R_0^2}{1-2R_0^2\Omega(R_H/R_X)}, \qquad (2)$$

where Ω is, as before, the function such that, if for short we put $R_H/R_X = \gamma$,

$$\Omega(\gamma) = \tfrac{1}{2}\gamma\sqrt{1+\gamma^2}+\tfrac{1}{2}\log\{\sqrt{1+\gamma^2}+\gamma\}. \qquad (3)$$

It is seen that (2) reduces to (11) of § 5.2.4 if we make various approximations.

The form of the integral depends on R_0. The motion of the axes does not concern us explicitly. A rough idea of the value of R_0 occurring in practice may be obtained by considering that at the vertex the sphere is not in rapid horizontal motion relative to the gun, and so R_0 is roughly equal to minus the naturalized speed of the wind relative to the ground at the vertex, that is

$$R_0 = -V_X, \text{ roughly}. \qquad (5)$$

The greatest wind-speed with which we have to contend may be put at 20 m/s., whereas the unit of speed is 40 m/s. for the pea-size of steel sphere, or 60 m/s. for the cherry size. Thus R_0 is not likely to exceed in absolute value $0\cdot5$ or $0\cdot33$ respectively. The larger value has been selected for illustration.

As some quadratures are inevitable, I have preferred to use them throughout. An approximate integral has been stepped out numerically for the worst case occurring in practice, namely, $R_0 = 0\cdot5$. The result is set out in the following table. The computation began with three rows headed $T = 0\cdot0$, $0\cdot05$, $0\cdot1$. The numbers in these three

* 'Handbook on Ballistics,' 1920. H.M. Stationery Office.

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rows were adjusted by repeated trial until the values of R_X and R_H at $T = 0\cdot05$ were midway between those at $T = 0\cdot0$ and $T = 0\cdot1$, and at the same time were such that the increases in R_X and R_H from $T = 0\cdot0$ to $T = 0\cdot01$ were equal respectively to $0\cdot1\ dR_X/dT$ and $0\cdot1\ dR_H/dT$ at the centre of this interval of T, when dR_X/dT and dR_H/dT were given by

$$dR_X/dT = -R_X\left|\sqrt{R_X^2+R_H^2}\right|, \qquad (6)$$

$$dR_H/dT = -R_H\left|\sqrt{R_X^2+R_X^2}\right|-1. \qquad (7)$$

After this troublesome beginning the calculation proceeded more easily by the "step over" method.*

Next, there was made a calculation of just the same type, beginning in the same way except that all the differences of T were doubled. The error of the integral is proportional to the square of the co-ordinate difference if the latter is both centred and small enough.† Assuming this to be the case, the two finite-difference integrals provide us with a means of extrapolating to the infinitesimal integral. This has been done and the numbers shown in the following table have been corrected in this way. The correction nowhere amounts to more than 6 in the last place of digits printed.

Exact integral of the equations of motion (6) and (7), applicable to motion in a wind which is independent of height when $R_X = 0\cdot500$ at the vertex.

TABLE IX.

T.	R_X.	R_H.	Z.	$\int_0^T R_X\,dT$.	$\frac{R_X}{\sec T}$ when T negative. $\frac{R_X}{\text{sech } T}$ when T positive.
−0·6	0·768	0·770	0·209	−0·364	0·634
−0·5	0·695	0·602	0·140	−0·291	0·610
	0·652		0·100		
−0·4	0·638	0·457	0·087	−0·224	0·588
−0·3	0·594	0·329	0·048	−0·163	0·567
−0·2	0·557	0·212	0·021	−0·105	0·546
−0·1	0·526	0·103	0·005	−0·051	0·523
0·0	0·500	0·000	0·000	0·000	0·500
0·1	0·476	−0·098	0·005	0·049	0·478
0·2	0·454	−0·191	0·019	0·095	0·463
0·3	0·432	−0·279	0·043	0·140	0·452
0·4	0·409	−0·362	0·075	0·181	0·442
	0·379		0·100		
0·5	0·387	−0·439	0·115	0·221	0·436
0·6	0·364	−0·510	0·163	0·259	0·431
0·7	0·341	−0·575	0·217	0·294	0·428

The numbers in the last column may be ignored until we come to § 5.4.

* For further particulars of which *see* 'Weather Prediction by Numerical Process,' Chapter 7/2. Cambridge University Press.

† *Vide* L. F. RICHARDSON, 'Phil. Trans.,' A, Vol. 210, § 1.2 (1910).

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As by hypothesis the naturalized thickness of the calm is to be $Z = 0\cdot2$ it is found, by interpolation in the preceding table, that the sphere must have entered the wind at

$$T_1 = -0\cdot588\,;\quad R_{X1} = 0\cdot758\,;\quad R_{H1} = 0\cdot749\,;\quad \int_0^{T_1} R_X\,dT = -0\cdot355,$$

and have left the wind again with

$$T_2 = +0\cdot670\,;\quad R_{X2} = 0\cdot348\,;\quad R_{H2} = -0\cdot556\,;\quad \int_0^{T_2} R_X\,dT = 0\cdot284.$$

These numbers are confirmed by the fact that they satisfy the relations (2), (3) above.

The Motion in the Calm.—May we apply to this region the simple theory in which unity is put in place of $\sqrt{\{1+(R_X/R_H)^2\}}$?

The largest values of R_X/R_H occur either (i) just under the interface where R_H is smallest, or (ii) at the muzzle. Consider the former. There, R_X can be found from the preceding table (IX) as soon as we know the jump in R_X at the interface, namely, $\pm V_X$. We already know that $V_X = -R_0$ roughly. If so, the sphere just before entering the wind has $R_X = 0\cdot758-0\cdot500 = 0\cdot258$, $R_H = 0\cdot749$, while just after leaving the wind it has $R_X = 0\cdot348-0\cdot500 = -0\cdot152$, while $R_H = -0\cdot556$.

In these two cases $\sqrt{\{1+(R_X/R_H)^2\}} = 1\cdot06$ and $1\cdot04$ severally.

The region of this error must be quite a thin one, as R_H increases downwards.

The error at the muzzle cannot be estimated until we have found β.

So let us proceed, putting $\sqrt{\{1+(R_X/R_H)^2\}}$ equal to unity. We may then apply the theory of § 4 with this proviso, that the height of the vertex to be used in that theory is not the actual height attained in the wind, but the height which the sphere would have risen to, or descended from, if it had not been for the large values of R_X in the wind. Only thus shall we obtain the relation between S and T which allows us to use the tabulated values of J. The formulæ for vertical motion given in § 4.3 are used here. Thus in the ascent the sphere proceeds as if to a vertex Z_1 above the interface, where

$$Z_1 = \tfrac{1}{2}\log\left(1+R_{H1}^2\right) = 0\cdot223,$$

while in the descent the vertical motion corresponds to a different vertex, Z_2, above the interface, where

$$Z_2 = \tfrac{1}{2}\log\left(1-R_{H2}^2\right) = 0\cdot185.$$

With this proviso the theory is an extension of that given in § 5.2.3 above. In general

$$\dot{X} = \text{const.}\ e^{-S}. \qquad (1)$$

But on leaving the muzzle at $S = -(C+Z_1)$ the sphere has $\dot{X} = \beta U$. The arbitrary constant is thus determined so that throughout the ascent

$$\dot{X} = \beta U e^{-S-C-Z_1}. \qquad (2)$$

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The naturalized horizontal distance travelled during the ascent through the calm is, therefore,

$$X_{\text{ascent}} = \int_{S=-C-Z_1}^{S=-Z_1} \dot{X}\, dT = \beta U e^{-C-Z_1} \int_{S=-C-Z_1}^{S=-Z_1} e^{-S}\, dT = \beta U e^{-C-Z_1} \{J_{-Z_1} - J_{-C-Z_1}\}. \quad (3)$$

The sphere then enters the wind in which it acquires changes of both position and speed relative to the ground. Let the naturalized speeds on crossing the interface be

in rising: $\dot{X}_1$ relative to ground, R_{X1} relative to air;
in falling: $\dot{X}_2$,, ,, R_{X2} ,, ,,

Then, as under all circumstances by the principle of superposition of velocities

$$R_X = \dot{X} - V_X, \quad (4)$$

it follows that

$$R_{X1} = \dot{X}_1 - V_X, \quad (5)$$

$$R_{X2} = \dot{X}_2 - V_X, \quad (6)$$

whence by subtraction $\dot{X}_2 - \dot{X}_1$ which we require, is equal to $R_{X2} - R_{X1}$ which we have already found from the theory of the wind-region . . . . (7)

But putting in (2) $S = -Z_1$ we find that

$$\dot{X}_1 = \beta U e^{-C}. \quad (8)$$

And so by (5) and (7) we obtain V_X as a function of the unknown β and known quantities, thus

$$V_X = \beta U e^{-C} - R_{X1}. \quad (9)$$

The naturalized horizontal distance travelled in the wind relative to the ground is $\int \dot{X}\, dT$ taken between the limits T_1 for entry and T_2 for exit. This distance is, therefore, by (4)

$$\int_{T_1}^{T_2} (R_X + V_X)\, dT = \int_{T_1}^{T_2} R_X\, dT + V_X (T_2 - T_1) = \int_{T_1}^{T_2} R_X\, dT + (\beta U e^{-C} - R_{X1})(T_2 - T_1). \quad (10)$$

Now, in the descent by (7) and (8)

$$\dot{X}_2 = R_{X2} - R_{X1} + \beta U e^{-C}. \quad (11)$$

And so, as (1) holds in general, the constant appropriate to the descent through the calm is such that

$$\dot{X}_2 = \text{const.}\ e^{-Z_2}$$

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whence, throughout the descent

$$\dot{X} = e^{Z_2}\{R_{X2}-R_{X1}+\beta U e^{-C}\}\, e^{-S}. \quad \quad (12)$$

and the naturalized horizontal displacement during the descent through the calm is

$$X_{\text{descent}} = \int_{S=Z_2}^{S=Z_2+C} \dot{X}\, dT = e^{Z_2}\{R_{X2}-R_{X1}+\beta U e^{-C}\}\{J_{Z_2+C}-J_{Z_2}\}. \quad \quad (13)$$

Since the returning ball hits the gun, the whole horizontal displacement is zero. Thus, collecting the parts of this displacement given by (3), (10) and (13) we have

$$0 = \beta U e^{-C-Z_1}\{J_{-Z_1}-J_{-C-Z_1}\} + \int_{T_1}^{T_2} R_X \cdot dT + (\beta U e^{-C} - R_{X1})(T_2-T_1)$$
$$+ e^{Z_2}\{R_{X2}-R_{X1}+\beta U e^{-C}\}\{J_{Z_2+C}-J_{Z_2}\}, \quad . \quad (14)$$

which is the equation to find β. It is of the simple form $0 = m_1\,\beta + m_2$ where m_1 and m_2 are known numbers.

When β has been found by this equation then V_X follows by (9).

To agree with our specimen observation* take $C = 1\cdot 2$. It is then found from (14) that $\beta = 0\cdot 171$. Thus near the muzzle we have regarded $\surd\{1+(0\cdot 17)^2\} = 1\cdot 014$ as unity, which is quite a small error. By (9) the naturalized wind-speed is $V_X = -0\cdot 551$ which differs noticeably from $-R_0 = -0\cdot 500$.

This completes a fairly exact solution.

It may now be used to test the simpler approximation of § 5.2. Taking the value just found for β, namely, $\beta = 0\cdot 171$ together with $L = 1\cdot 4$, $Z = 0\cdot 2$, and treating these by the method of § 5.2 just as if they were observations, it is found that $V_X = -0\cdot 61$. The true value is $V_X = -0\cdot 55$; so that the approximate result is 11 per cent. high in this extreme example. Internal evidence shows that the errors diminish rapidly as V_X and Z diminish, so that the approximate theory of § 5.2 may be considered as efficient for all ordinary winds for which $V < 0\cdot 5$ if the naturalized vertical penetration Z is cut down from $0\cdot 2$ to $0\cdot 1$.

§ 5.4. *Correction to Compensate for the Approximation* $\surd\{1+(R_X/R_H)^2\} = 1$ *when the Wind Distribution is Moderately General.*

The theory of § 5.2, § 5.3 is sufficiently correct in itself, and applies only to a special distribution of wind; but let us now suppose that the wind distribution is more general, and that the observations have been first reduced by the process of § 4, which puts $\surd\{1+(R_X/R_H)^2\} = 1$; and let us now seek to adjust this first approximation.

It is impossible to determine the correction accurately without making for each wind distribution elaborate quadratures similar to those made for the preceding test. But

* Shown in the diagram on p. 367.

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what we can do is to find a rough value of the correction which will at least tell us when, as often, it is negligible.

There are two parts of the trajectory in which R_X/R_H is apt to be troublesomely large, namely (1) just below the vertex, as we have seen; and (2) at the beginning of the flight. In a uniform wind, calculation shows that the two error-producing parts join one another inseparably. Uniform winds, however, are seldom observed, the actual wind velocity usually increasing aloft. The distribution then tends rather towards the type which we have considered in detail—wind above and calm below. Thus, in practice the error which we are exploring would appear to be confined to the part of the path close before and after the vertex, a region in which R_X is greater than or comparable with R_H.

But near the vertex the vertical motion goes on almost as if *in vacuo*, so that $R_H = -T$. That being so, the error-producing portion of the path will be traversed in a naturalized time comparable with R_X during the ascent, and in a like time in descending.

Now at the vertex, where the error is greatest, it amounts to neglecting a naturalized acceleration of $-R_0^2$ where R_0 is the value of R_X at the vertex. This may be seen by considering the equation of motion

$$d^2X/dT^2 = -R_X|\sqrt{(R_X^2+R_H^2)}|,$$

for in the "linear" theory the second member is taken as if it were zero at the vertex.

A neglect of an acceleration during a short time is equivalent to the neglect of the impulsive velocity which is the product of the time and the acceleration. So it looks as if we may regard the correction as an impulsive change in the R_X occurring at the vertex, proportional to R_0^3 and further probably of the order of R_0^3.

We have already—in § 5.2 above—found the tilt required to balance any known impulsive change of R_X at the vertex.

We may examine the correcting change in R_X more closely when the wind is independent of height near the vertex. For then $d^2X/dT^2 = dR_X/dT$ so that the equations of motion are

$$dR_X/dT = -R_X|\sqrt{(R_X^2+R_H^2)}|,$$
$$dR_H/dT = -R_H|\sqrt{(R_X^2+R_H^2)}|-1.$$

On solving these, with neglect of R_X under the square root, it is found that

$\left\{\begin{array}{l}\text{in the ascent } R_X/\sec T\\ \text{in the descent } R_X/\text{sech}\, T\end{array}\right\}$ is a constant equal to R_0.

Now, if we take an exact integral such as that in the table on p. 376 made without neglecting R_X under the square root, and work out for successive points on the trajectory the true values of $R_X/\sec T$ in the ascent and of $R_X/\text{sech}\, T$ in the descent, it is found that these parameters vary. They may be seen in the last column of Table IX on p. 376. In other words, R_0 changes as if by a succession of impulses. Now, if we seek to find the horizontal layer of air in which most of this variation of R_0 occurs, it is found from the aforesaid table, and confirmed and extended by other calculations here omitted

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for brevity, that the variation is rapid in a thin layer just below the vertex, and slows down without stopping in the lower layers. That is in a uniform wind. But in a typical wind whose speed is less below, the slowing down of the variation of R_0 will be more marked, so that it seems permissible for rough purposes to regard the change in R_0 as an impulse concentrated at the vertex.

We have already seen that this change in R_0, which we may call ΔR_0, is probably proportional to R_0^3. This idea was confirmed by calculations which are rather long and so are omitted here, which were made for the two cases $R_0 = 0\cdot5$, $0\cdot2$. It was thus found that

$$\Delta R_0 = 2R_0^3, \text{ roughly.}$$

Again, the horizontal speed of the ball relative to the gun is small at the vertex, so that $R_0 = -V_0$ roughly, where V_0 is the naturalized wind-speed at the vertex. Consequently

$$\Delta R_0 = -2V_0^3 \text{ approximately.}$$

The amount of the correction to be added to β is, therefore, by § 5.2.3 equal to

$$\frac{\pm 2V_0^3}{Ue^{-L}(1-J_{-L}/J_L)} \text{ approximately,}$$

the denominator of which has been tabulated on p. 370 above. The sign of the correction is best determined by the consideration that in putting $\sqrt{\{1+(R_X/R_H)^2\}} = 1$ we have reduced the effect of the wind, so that the tilts corrected for use in the approximate theory, should be less in absolute magnitude than the observed tilts, provided that the wind does not reverse its direction aloft.

This correction has been extensively applied to observations in the following manner. Firstly, the "raw" tilts are used with the approximate general theory of § 4 to find the wind distribution. Thus V_0, which is a resultant of horizontal components, is found for each of the vertices. The tilts are then corrected and reintroduced into the general approximate theory to obtain the winds.

The correction is seldom more than 10 per cent., sometimes less than 1 per cent. If the sphere, after rising through calm, just entered a wind layer, the correction would be too big. These distributions of wind are, therefore, not computed by the general theory, which puts $\sqrt{\{1+(R_X/R_H)^2\}} = 1$ at the vertex, but instead by the special theory of § 5.2 which does not make this approximation. The boundary between the regions served by these two theories seems to be at $Z = 0\cdot2$ or thereabouts.

Illustration on § 5.4.—Turning back to the observation reduced in § 4.7, it is seen that the linear theory gave wind components of $2\cdot5$ and $5\cdot7$ m/s. at $L = 1\cdot8$. The resultant is, therefore, $6\cdot2$ m/s., which expressed in the "natural" unit of $59\cdot9$ m/s. is $V = 0\cdot103$. So regarding $L = 1\cdot8$ as the position of one of the vertices, $V_0 = 0\cdot103$. The correction to β is

$$\pm 2V_0^3/\{Ue^{-L}(1-J_{-L}/J_L)\}.$$

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The denominator of this is found from the table on p. 370 to be 2·72. Hence the correction to β comes to

$$\pm 2\,(0\cdot 103)^3/2\cdot 72 = 0\cdot 0008.$$

This is only about $1\frac{1}{3}$ per cent. of the observed resultant β, and so has been neglected.

§ 6. Deviations from the Square Law of Resistance.

When n in the square-law theory is no longer a constant, but varies with r_H, it would still be possible to calculate functions analogous to j and f, and to use them to transform the integral equation approximately into a set of linear algebraic equations convenient for the use of the observer. But many of these calculations would have to be made by toilsome quadratures (once for all), and so it would be well, before beginning them, to be sure of the exact form of the law of force. As yet there is no certainty.* Furthermore, the law of force apparently depends upon the diameter of the sphere, so that a convenient size should first be agreed upon.

No complication in the observer's work is to be expected from these complications in the law of force, but merely that he should be provided with a slightly different set of numerical constants.

Reference should here be made to the theory of § 3, because it is true, with the approximations indicated, for any law of force.

§ 7. Personal.

For the opportunity to carry out this research, which was done as an official duty for the Meteorological Office in 1919 to 1920, I am indebted to the interest taken in the matter by Sir Napier Shaw, F.R.S., who was then Director, and by Mr. W. H. Dines, F.R.S., on whose field many experiments were made.

§ 8. List of Symbols, with Numbers of Pages on which they are Defined.

β, β_X, β_Y, t', p. 346; t, x, y, h, p. 347; s, r_X, r_Y, r_H, r, g, t_l, t_{-l}, l, a, b, p. 348; c, u, v_X, p. 349; k, p. 351; capital Roman letters, see p. 351, and the corresponding small letter; $\ddot{x}$, $\ddot{y}$, $\ddot{h}$, $\dot{h}$, $\dot{s}$, p. 354; $\dot{x}$, p. 355; j, p. 357; f, p. 358; ψ, p. 359; numerical suffixes, pp. 361, 362; Z, p. 365; R_{X1}, R_{X2}, $\dot{X}_1$, $\dot{X}_2$, p. 368; R_0, p. 369, line 3; ϕ, μ, p. 370; Ω, α, p. 372; equations (12), (13); C, p. 375; γ, p. 375; m_1, m_2, p. 379; ΔR_0, p. 381.

* It is hoped that a study of "The Aerodynamic Resistance of Spheres shot Upward to Measure the Wind" may be published later elsewhere.